

Solution Sheet – 2008

4th – 5th Grades

1. When the six-digit number $3456N7$ is divided by 8, the remainder is 5. List all possible values of the digit N .

Answer: 3, 7

Solution: If number 345607 ($N = 0$) is divided by 8, the remainder is 7 (do it!) If we increase N by 1, the number would increase by 10, so the remainder would “increase” by 2. So if $N = 1$, the remainder would be 9, but it could be divided further by 8, and the actual remainder is 1; if $N = 2$, the remainder is 3; if $N = 3$, the remainder is 5; if $N = 4$, the remainder is 7 again; if $N = 5$, the remainder is 1 again; if $N = 6$, the remainder is 3 again; if $N = 7$, the remainder is 5 again; if $N = 8$, the remainder is 7 again; if $N = 9$, the remainder is 1 again. As we see, remainder 5 corresponds to $N = 3$ and $N = 7$.

Alternate Solution: The condition “The number $3456N7$ has the remainder 5 if divided by 8” is the same as “The number $3456N2$ is divisible by 8” that is the same as “The number $6N2$ is divisible by 8”. Since 600 is divisible by 8 ($600 = 75 \times 8$), number $N2$ should be divisible by 8. Now we can check all 1- and 2-digit multiples of 8 (0, 8, 16, 24, 32, 40, 48, 56, 64, 72, 80, 88, 96) and find those that have 2 as their last digit (32 and 72 only). Therefore $N = 3$ or $N = 7$.

Another Alternate Solution: We can just divide all 10 numbers 345607 , 345617 , ..., 345697 by 8 (do it!), check the remainders, and find the numbers that correspond to the remainder 5.

2. How many positive factors does number 72 have?

Answer: 12

Solution: Let’s consider prime decomposition of the number 72: $72 = 9 \times 8 = (3 \times 3) \times (2 \times 2 \times 2)$. To compose a positive factor of 72, we need to select the prime factor 3 0, 1 or 2 times (3 choices); we also need to select the prime factor 2 0, 1, 2, or 3 times (4 choices); and then we need to multiply all these selected prime factors ($3 \times 4 = 12$ choices). So number 72 has 12 positive factors. As always we assume that if we select no prime factors and multiply “them”, the result would be 1.

Alternate Solution: We will check some small positive whole numbers (starting from 1) whether they are factors of 72 or not. Whenever we find a factor, we will also write down another “brother” factor (the product of both of them would be 72). 1 is the factor of 72, $1 \times 72 = 72$, so we already know two “brother” factors – 1 and 72. $2 \times 36 = 72$ (two more factors), $3 \times 24 = 72$ (two more factors), $4 \times 18 = 72$ (two more factors), 5 is not a factor of 72 (the last digit of the number 72 is neither 5 nor 0), $6 \times 12 = 72$ (two more factors), 7 is not a factor of 72 ($72 \div 7 = 10$ R2), $8 \times 9 = 72$ (two more factors). Since 8 and 9 are consecutive whole numbers, we do not need to continue the factor search: if we find any factor of 72 greater than 8, its “brother” factor would be less than 9, but we have already listed all the factors of 72 that are less than 9, so we have found all positive factors of the number 72. Altogether we have listed 12 factors.

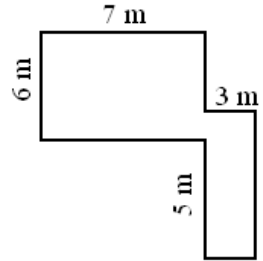
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3. If 1 Tom is equal to 7 Fan, 2 Harry is equal to 1 Husk, and 14 Tom is equal to 1 Husk, then how many Fan make up a Harry?

Answer: 49

Solution: Since 1 Husk is equal to 2 Harry and 1 Husk is equal to 14 Tom, 2 Harry is equal to 14 Tom, therefore 1 Harry is equal to 7 Tom. But 1 Tom is equal to 7 Fan, therefore 1 Harry is equal to $7 \times 7 = 49$ Fan.

4. What is the perimeter (in meters) of the figure shown? All angles are right angles.



Answer: 42

Solution: The perimeter of the figure shown is equal to the total length of its left side (2 parts), right side (2 parts), top side (2 parts), and bottom side (2 parts). From the figure we know that the total length of the figure’s left side is $5m + 6m = 11m$, and the total length of its top side is $7m + 3m = 10m$. But the total length of its right side is the same as the total length of its left side (11m), and the total length of its bottom side is the same as the total length of its top side (10m). Therefore the perimeter of the figure is equal to $11m + 11m + 10m + 10m = 42m$.

5. Bay Street has between 2 and 15 houses, numbered 1, 2, 3, and so on. Mr. Sullivan lives in one of the houses. The sum of all the house numbers less than his equals the sum of all the house numbers greater than his. How many houses are there on Bay Street?

Answer: 8

Solution: The sum of all the house numbers on Bay Street (let’s call it *A*) is equal to the sum of all the house numbers less than Mr. Sullivan’s house number, plus Mr. Sullivan’s house number, and plus the sum of all the house numbers greater than Mr. Sullivan’s house number. And it is the same, as Mr. Sullivan’s house number plus twice the sum of all the house numbers less than Mr. Sullivan’s house number (let’s call this *B*). Let’s create and fill in the following tables:

Number of houses on Bay Str.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
<i>A</i>	1	3	6	10	15	21	28	36	45	55	66	78	91	105	120

Mr. Sullivan’s house number	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
<i>B</i>	1	4	9	16	25	36	49	64	81	100	121	144	169	196	225

As we can see, there are only two numbers that appear in both rows *A* (first table) and *B* (second table). Number 1 means that there is only 1 house on Bay Street – Mr. Sullivan’s house, but problem states that Bay Street has between 2 and 15 houses. Number 36 means that there are 8 houses on Bay Street, and Mr. Sullivan lives in the house 6. The sum of all the house numbers less than his equals $1 + 2 + 3 + 4 + 5 = 15$, and the sum of all the house numbers greater than his equals $7 + 8 = 15$.

6. Let’s define an 8-digit positive whole number *abcdefgh* (*a, b, c, d, e, f, g, h* – its digits, $a > 0$) as “beautiful” if numbers *aceg* and *bdfh* are the same (for example, 22000088.) Find the number of “beautiful” numbers.

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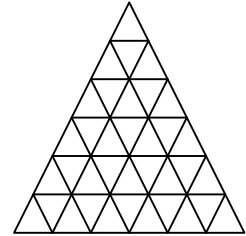
Answer: 9000

Solution: First (from the left) digit a could be any digit from 1 to 9 (9 choices), second digit b should be the same as a (1 choice), third digit c could be any digit from 0 to 9 (10 choices), fourth digit d should be the same as c (1 choice), and so force (10 choices for e , 1 choice for f , 10 choices for g , 1 choice for h). Now we can count that there are $9 \times 1 \times 10 \times 1 \times 10 \times 1 \times 10 \times 1 = 9000$ “beautiful” numbers (by using the so-called “multiplication rule”).

7. How many triangles are in this figure? Count all triangles you can see without drawing additional lines.

Answer: 78

Solution: Let’s count triangles by their size and orientation. We assume that length of each side of the big triangle is 6. First, let’s consider only triangles that have the same orientation as the big triangle. Their sides could have length 6 (1 triangle), length 5 (1 + 2 = 3 triangles), length 4 (1 + 2 + 3 = 6 triangles), length 3 (1 + 2 + 3 + 4 = 10 triangles), length 2 (1 + 2 + 3 + 4 + 5 = 15 triangles), and length 1 (1 + 2 + 3 + 4 + 5 + 6 = 21 triangles), in total 56 triangles. Then let’s consider all opposite-oriented triangles. Their sides could have length 3 (1 triangle), length 2 (1 + 2 + 3 = 6 triangles), and length 1 (1 + 2 + 3 + 4 + 5 = 15 triangles), in total 1 + 6 + 15 = 22 triangles. Therefore, there are 56 + 22 = 78 triangles in the figure shown.



8. There are 6 coins that look the same, each of them in its own clear plastic bag (bags are labeled using letters A, B, C, D, E, F and have the same weight.) 3 of these coins are genuine, having the same weight, and 3 others are fake, also having the same weight, but a fake coin is lighter than a genuine one. There is also a correctly functioning balance without any reference weights and without a scale. What is the least number of weigh attempts an expert should make to surely identify all genuine coins? The expert does not have any additional coins.

Answer: 3

Solution: There are 20 possible distributions of 3 genuine coins among the given 6 coins: $ABC, ABD, ABE, ABF, ACD, \dots, CDE, CDF, CEF, DEF$ (list them all!) Whenever the expert uses the balance to compare some coins (we call it weigh attempt) there are just 3 possible outcomes: pans are in balance, the left pan is heavier than the right one, or the right pan is heavier than the left one. So after the first weigh attempt (regardless of what coins were on each balance pan) in the worst case out of 20 initial possible distributions of genuine coins there would be at least 7 distributions that satisfy the result of this weigh attempt ($6 \times 3 = 18 < 20$). And after the second weigh attempt in the worst case out of these at least 7 distributions of genuine coins there would be at least 3 distributions that also satisfy the result of this weigh attempt ($2 \times 3 = 6 < 7$). Therefore the expert would not be able surely identify all 3 genuine coins, using just 2 weigh attempts.

Let’s show (using the flow-chart below) how the expert could surely identify all 3 genuine coins using 3 weigh attempts. We will always put exactly one coin on each of the balance pans, so whatever coin is heavier it would be genuine one, whatever coin is lighter it would be fake one, and if the pans are in balance the coins would be both genuine or fake.

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Attempt 1: $A ? B$

Outcome 1: $A = B$

Attempt 2: $B ? C$

Outcome 1.1: $B = C$

Attempt 3: $C ? D$

Outcome 1.1.1: $C = D$ – impossible (4 coins with the same weight)

Outcome 1.1.2: $C > D$ – A , B , and C are the 3 genuine coins

Outcome 1.1.3: $C < D$ – D , E , and F are the 3 genuine coins

Output 1.2: $B > C$ – both A and B are genuine coins, C is a fake coin

Attempt 3: $D ? E$

Outcome 1.2.1: $D = E$ – both D and E are fake coins, F is the last genuine coin

Outcome 1.2.2: $D > E$ – D is the last genuine coin

Outcome 1.2.3: $D < E$ – E is the last genuine coin

Output 1.3: $B < C$ – it is similar to the Outcome 1.2: $B > C$ above

Outcome 2: $A > B$ – A is a genuine coin, B is a fake coin

Attempt 2: $C ? D$

Outcome 2.1: $C = D$

Attempt 3: $D ? E$

Outcome 2.1.1: $D = E$ – impossible (4 coins with the same weight)

Outcome 2.1.2: $D > E$ – C and D are the two remaining genuine coins

Outcome 2.1.3: $D < E$ – E and F are the remaining last genuine coins

Outcome 2.2: $C > D$ – C is a genuine coin, D is a fake coin

Attempt 3: $E ? F$

Outcome 2.2.1: $E = F$ – impossible (4 coins with the same weight)

Outcome 2.2.2: $E > F$ – E is the last genuine coin

Outcome 2.2.3: $E < F$ – F is the last genuine coin

Outcome 2.3: $C < D$ – it is similar to the Outcome 2.2: $C > D$ above

Outcome 3: $A < B$ – it is similar to the Outcome 2: $A > B$ above

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6th – 7th Grades

1. My watch *bips* every minute. Jim's watch *bops* every 62 seconds. If our *bip* and *bop* coincide at 12:00 pm, what is the next time our *bip* and *bop* coincide?

Answer: 12:31 pm

Solution: Since 1 min = 60 sec, we need to figure out the smallest number of seconds that is divisible by both 60 sec and 62 sec. In other words, we need to find LCM (The Least Common Multiple) of 60 and 62. $60 = 2 \times 2 \times 3 \times 5$, $62 = 2 \times 31$, 2, 3, 5, and 31 are primes, so $LCM(60, 62) = 2 \times 2 \times 3 \times 5 \times 31 = 60 \times 31$, but 60×31 sec = 31 min, so their *bip* and *bop* coincide again in 31 min after 12:00 pm, and continue coincide every 31 min.

2. Find the sum of the first 100 terms of the series: $\underbrace{1 + 2 + 5 + 6 + 9 + 10 + 13 + 14 + \dots + 197 + 198}_{100 \text{ numbers}}$.

Answer: 9950

Solution:

$$1 + 2 + 5 + 6 + 9 + 10 + 13 + 14 + \dots + 197 + 198 = (1 + 198) + (2 + 197) + (5 + 194) + (6 + 193) + \dots + (98 + 101) = 199 \times 50 = 9950.$$

3. Which number is greater: 2008^{100} or $2007^{100} + 2007^{99}$?

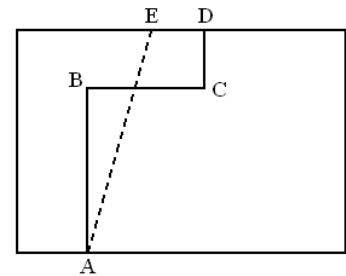
Answer: 2008^{100}

Solution: $2007^{100} + 2007^{99} = 2007^{99}(2007 + 1) = 2007^{99} \times 2008 < 2008^{99} \times 2008 = 2008^{100}$.

4. Two pieces of rectangular land are separated by the line $ABCD$, as shown on the diagram. $AB = 30$ m, $BC = 24$ m, and $CD = 10$ m. AB , BC , and CD are parallel to the sides of the rectangle. We want to straighten the border line by the straight line AE , so that the areas of both land pieces would not change. What is the distance ED (in meters)?

Answer. 12

Solution: Let's extend segment AB until intersection with the top side of the big rectangle, let F be the point of intersection. Before the border change, the left piece of land contained the rectangle to the left of AF and rectangle $BCDF$; after the border change, the left piece of land would contain the same rectangle to the left of AF , and triangle AEF . In order for the left piece of land to have the same area, the area of the rectangle $BCDF$ ($24 \text{ m} \times 10 \text{ m} = 240 \text{ m}^2$) should be the same as the area of the triangle AEF ($AF \times FE / 2$). Since $AF = AB + CD = 40$ m, $20 \times FE = 240$, $FE = 12$ m. Therefore $ED = FD - FE = BC - FE = 24 - 12 = 12$ m.



5. Based on a survey conducted in a group of 13-year olds, it turned out that they watch TV for an average of 50 minutes every day. The average time the boys in this group watch TV was 45 minutes per day, and the average time the girls in this group watch TV was 65 minutes per day. What is the ratio of the numbers of the boys and the girls in this group?

Answer: 3:1

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Solution: Let B be the number of the boys in the group surveyed, and G be the number of the girls in this group. Based on the survey, B boys watch TV in total $45B$ min per day, and G girls watch TV in total $65G$ min per day. Since there are only boys and girls in this group and nobody else, all of them watch TV in total $45B + 65G$ per day. On the other hand, based on the survey, they watch TV in total $50(B + G)$ min per day, so $45B + 65G = 50(B + G)$, $15G = 5B$, $B = 3G$. This means that there are 3 times more boys than girls in the group surveyed.

6. Let's define an 8-digit positive whole number $abcdefgh$ (a, b, c, d, e, f, g, h – its digits, $a > 0$) as “beautiful” if at least two of its neighboring digits are the same (for example, 20080316 or 20000000.) Find the number of “beautiful” numbers.

Answer: 46953279

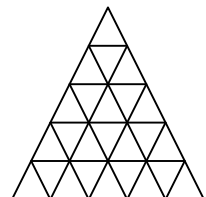
Solution: We will count the number of “beautiful” numbers (B) as the total number of 8-digit positive whole numbers (T) minus the number of “non-beautiful” 8-digit positive whole numbers (NB). In an 8-digit positive whole number, first (from the left) digit a could be any digit from 1 to 9 (9 choices), second digit b could be any digit from 0 to 9 (10 choices), third digit c could be any digit from 0 to 9 (10 choices), and so force (10 choices for each of digits d, e, f, g , and h). Now we can count that $T = 9 \times 10^7 = 90000000$ (by using the so-called “multiplication rule”). In a “non-beautiful” 8-digit positive whole number, first (from the left) digit a could be any digit from 1 to 9 (9 choices), second digit b could be any digit from 0 to 9 except a (9 choices), third digit c could be any digit from 0 to 9 except b (9 choices), and so force (9 choices for each of digits d, e, f, g , and h). We can count that $NB = 9^8 = 81^4 = 6561^2 = 43046721$ (by using the same “multiplication rule” again). Finally, $B = T - NB = 90000000 - 43046721 = 46953279$.

Alternate Solution: An 8-digit positive whole number $abcdefgh$ is “beautiful” if and only if one of the following disjoint (not happening together) cases would happen: either $a = b$; or $a \neq b$, but $b = c$; or $a \neq b$, $b \neq c$, but $c = d$; and so force. For the case $a = b$, first (from the left) digit a could be any digit from 1 to 9 (9 choices), second digit b should be the same as a (1 choice), third digit c could be any digit from 0 to 9 (10 choices), and so force (10 choices for each of digits d, e, f, g , and h). And we can count all 8-digit positive whole numbers $abcdefgh$ that satisfy condition $a = b$: $9 \times 1 \times 10^6 = 9 \times 10^6$ (by using the so-called “multiplication rule”). For the case $a \neq b$, but $b = c$, first (from the left) digit a could be any digit from 1 to 9 (9 choices), second digit b could be any digit from 0 to 9 except a (9 choices), third digit c should be the same as b (1 choice), fourth digit d could be any digit from 0 to 9 (10 choices), and so force (10 choices for each of digits e, f, g , and h). And we can count all 8-digit positive whole numbers $abcdefgh$ that satisfy condition $a \neq b$, but $b = c$: $9 \times 9 \times 1 \times 10^5 = 9^2 \times 10^5$ (by using the same “multiplication rule” again). Similarly, we get $9^3 \times 10^4$, $9^4 \times 10^3$, $9^5 \times 10^2$, $9^6 \times 10$, 9^7 “beautiful” numbers for the remaining conditions. Now we can use the so-called “addition rule” to find the total number of “beautiful” numbers: $9 \times 10^6 + 9^2 \times 10^5 + 9^3 \times 10^4 + 9^4 \times 10^3 + 9^5 \times 10^2 + 9^6 \times 10 + 9^7 = 46953279$.

7. How many quadrilaterals are in this figure? Count all quadrilaterals you can see without drawing additional lines.

Answer: 243

Solution: Let's count quadrilaterals by their shape, size, and orientation. We assume that length of each side of the big triangle is 5. All the quadrilaterals in the



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figure shown are either parallelograms or trapezoids. First, let's count parallelograms only. For each of the possible parallelograms, its sides are parallel to exactly two sides of the big triangle. There are exactly 3 ways to select 2 sides of the triangle, so we will count all parallelograms with sides parallel to non-horizontal sides of the big triangle, and multiply the result by 3. These parallelograms could contain 2 small triangles ($1 + 2 + 3 + 4 = 10$ choices), 4 small triangles ($2 \times (1 + 2 + 3) = 12$ choices), 6 small triangles ($2 \times (1 + 2) = 6$ choices), 8 small triangles (with sides 1 and 4) ($2 \times 1 = 2$ choices), 8 small triangles (with sides 2 and 2) ($1 + 2 = 3$ choices), and 12 small triangles ($2 \times 1 = 2$ choices), in total $3 \times (10 + 12 + 6 + 2 + 3 + 2) = 105$ parallelograms. Then, let's count trapezoids only. For each of the possible trapezoids, its bases are parallel to exactly one side of the big triangle. There are exactly 3 ways to select 1 side of the triangle, so we will count all trapezoids with bases parallel to the horizontal side of the big triangle, and multiply the result by 3. These trapezoids could have bases with length 1 and 2 ($1 + 2 + 3 + 4 = 10$ choices if base 2 is closer to the horizontal side of the big triangle than base 1, and $1 + 2 + 3 = 6$ choices if base 1 is closer to the horizontal side of the big triangle than base 2), length 1 and 3 ($1 + 2 + 3 = 6$ choices if base 3 is closer to the horizontal side of the big triangle than base 1, and just 1 choice if base 1 is closer to the horizontal side of the big triangle than base 3), length 1 and 4 ($1 + 2 = 3$ choices), length 1 and 5 (just 1 choice), length 2 and 3 ($1 + 2 + 3 = 6$ choices if base 3 is closer to the horizontal side of the big triangle than base 2, and $1 + 2 = 3$ choices if base 2 is closer to the horizontal side of the big triangle than base 3), length 2 and 4 ($1 + 2 = 3$ choices), length 2 and 5 (just 1 choice), length 3 and 4 ($1 + 2 = 3$ choices if base 4 is closer to the horizontal side of the big triangle than base 3, and just 1 choice if base 3 is closer to the horizontal side of the big triangle than base 4), length 3 and 5 (just 1 choice), and length 4 and 5 (just 1 choice), in total, $3 \times (10 + 6 + 6 + 1 + 3 + 1 + 6 + 3 + 3 + 1 + 3 + 1 + 1 + 1) = 138$ trapezoids. Therefore, altogether we have $105 + 138 = 243$ quadrilaterals in the figure shown.

8. There are 6 coins that look the same, each of them in its own clear plastic bag (bags are labeled using letters *A, B, C, D, E, F* and have the same weight.) At least 4 of these coins are genuine, having the same weight, and the others (may be none) are fake, also having the same weight, but a fake coin is lighter than a genuine one. There is also a correctly functioning balance without any reference weights and without a scale. What is the least number of weigh attempts an expert should make to surely identify all genuine coins? The expert does not have any additional coins.

Answer: 3

Solution: There are 15 possible distributions of 4 genuine coins (you can count distributions of 2 fake coins instead) among the given 6 coins: *ABCD, ABCE, ABCF, ..., CDEF* (list them all!); there are 6 possible distributions of 5 genuine coins (you can count distributions of 1 fake coin instead) among the given 6 coins: *ABCDE, ABCDF, ..., BCDEF* (list them all!); and there is just 1 possible distribution of 6 genuine coins among the given 6 coins: *ABCDEF*; so overall we have $15 + 6 + 1 = 22$ possible distributions of at least 4 genuine coins among the given 6 coins. Whenever the expert uses the balance to compare some coins (we call it weigh attempt) there are just 3 possible outcomes: pans are in balance, the left pan is heavier than the right one, or the right pan is heavier than the left one. So after the first weigh attempt (regardless of what coins were on each balance pan) in the worst case out of 22 initial possible distributions of genuine coins there would be at least 8 distributions that satisfy the result of this weigh attempt ($7 \times 3 = 21 < 22$). And after the second weigh attempt in the worst case out of these at least 8 distributions of genuine coins there would be at least 3 distributions that also

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satisfy the result of this weigh attempt ($2 \times 3 = 6 < 8$). Therefore the expert would not be able surely identify all genuine coins, using just 2 weigh attempts.

Let's show (using the flow-chart below) how the expert could surely identify all genuine coins using 3 weigh attempts. If we put exactly one coin on each of the balance pans, then whatever coin is heavier it would be genuine one, whatever coin is lighter it would be fake one, and if the pans are in balance the coins would be both genuine or fake.

Attempt 1: $AB ? CD$

Outcome 1: $AB = CD$

Attempt 2: $AC ? EF$

Outcome 1.1: $AC = EF$ – A , C , E , and F are genuine coins

Attempt 3: $A ? B$

Outcome 1.1.1: $A = B$ – both B and D are genuine coins

Outcome 1.1.2: $A > B$ – both B and D are fake coins

Outcome 1.1.3: $A < B$ – impossible (A is a genuine coin)

Output 1.2: $AC > EF$ – both A and C are genuine coins (otherwise we have at least 3 fake coins), either E or F (or both) is a fake coin, then both B and D are genuine coins (otherwise we have at least 3 fake coins)

Attempt 3: $E ? F$

Outcome 1.2.1: $E = F$ – both E and F are fake coins

Outcome 1.2.2: $E > F$ – E is a genuine coin, F is a fake coin

Outcome 1.2.3: $E < F$ – E is a fake coin, F is a genuine coin

Output 1.3: $AC < EF$ – both E and F are genuine coins (otherwise we have at least 3 fake coins), either A or C (or both) is a fake coin

Attempt 3: $B ? D$

Outcome 1.3.1: $B = D$ – both B and D are genuine coins (otherwise we have at least 3 fake coins), both A and C are fake coins

Outcome 1.3.2: $B > D$ – B is a genuine coin, D is a fake coin, A is a fake coin, C is a genuine coin

Outcome 1.3.3: $B < D$ – D is a genuine coin, B is a fake coin, C is a fake coin, A is a genuine coin

Outcome 2: $AB > CD$ – both A and B are genuine coins (otherwise we have at least 3 fake coins)

Attempt 2: $C ? D$

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Outcome 2.1: $C = D$ – both C and D are fake coins, both E and F are genuine coins

Outcome 2.2: $C > D$ – C is a genuine coin, D is a fake coin

Attempt 3: $E ? F$

Outcome 2.2.1: $E = F$ – both E and F are genuine coins

Outcome 2.2.2: $E > F$ – E is a genuine coin, F is a fake coin

Outcome 2.2.3: $E < F$ – E is a fake coin, F is a genuine coin

Outcome 2.3: $C < D$ – it is similar to the Outcome 2.2: $C > D$ above

Outcome 3: $AB < CD$ – it is similar to the Outcome 2: $AB > CD$ above

8th Grade

1. A palindrome is a whole number (first digit cannot be 0) that reads the same front to back or back to front (for example, 13231.) Compute the number of palindromes between 10 and 1000.

Answer: 99

Solution: Since 1000 is not a palindrome and we need to count only palindromes between 10 and 1000, our palindromes are either 2-digit or 3-digit positive whole numbers. In a 2-digit palindrome aa digit a could be any digit from 1 to 9 (9 choices), and in a 3-digit palindrome aba digit a could be any digit from 1 to 9 (9 choices), and digit b could be any digit from 0 to 9 (10 choices). Therefore, there are $9 + 9 \times 10 = 99$ palindromes between 10 and 1000.

2. Which number is greater: 2008^{100} or $2007^{100} + 2007^{99}$?

Answer: 2008^{100}

Solution: $2007^{100} + 2007^{99} = 2007^{99}(2007 + 1) = 2007^{99} \times 2008 < 2008^{99} \times 2008 = 2008^{100}$.

3. If p represents a positive prime number then what is the smallest p such that $p^{13} + 5p^{12}$ is a perfect cube?

Answer: 3

Solution: $p^{13} + 5p^{12} = p^{12}(p + 5)$ should be a perfect cube, but p^{12} is a perfect cube for any positive whole number p , so we need to find the smallest positive prime number p such that $p + 5$ is a perfect cube. If $p = 2$ (the smallest possible positive prime number), $p + 5 = 7$ is not a perfect cube. If $p = 3$ (the next smallest positive prime number), $p + 5 = 8$ is a perfect cube!

4. A confused bank teller transposed the dollars and cents when he cashed a check for Ms. Smith, giving her dollars instead of cents and cents instead of dollars. After buying a book for 9 dollars and 90 cents, Ms. Smith noticed that she had left exactly the amount of the original check. What was the amount of the check? Ms. Smith had no money before cashing the check.

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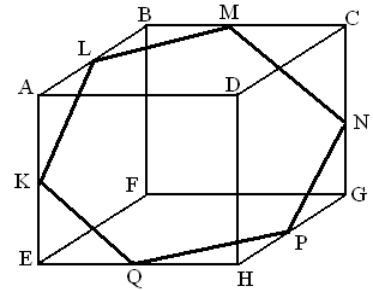
Answer: D dollars and $(D + 10)$ cents, $0 \leq D \leq 89, D \in \mathbf{Z}$

Solution: Let D be the amount of whole dollars in the original check (A is a positive whole number or 0), C be the amount of cents in this check (C is a whole number between 0 and 99). The teller gave Ms. Smith C dollars and D cents instead of D dollars and C cents, she had spent 9 dollars and 90 cents, and she had left exactly D dollars and C cents. Let's convert everything to cents (1 dollar = 100 cents) and get the following equation: $100C + D - 990 = 100D + C$, $99C = 99D + 990$, $C = D + 10$. So we can just re-phrase the problem: what could be the amount of the original check, if it contains 10 less dollars than cents? Since number of dollars D could not be negative, and number of cents C is less than 100, the answer is the following: the amount of the original check could be D dollars and $(D + 10)$ cents, where D is a whole number between 0 and 89.

5. $ABCDEFGH$ is a cube with side length 1. K, L, M, N, P, Q are the midpoints of AE, AB, BC, CG, GH, HE respectively. Compute the area of the regular hexagon $KLMNPQ$.

Answer: $3\sqrt{3}/4$

Solution: $AK = AL = 0.5$, $\angle LAK = 90^\circ$, so LAK is an isosceles right triangle, and $KL = AK\sqrt{2} = \sqrt{2}/2$. Let O be the center of the regular hexagon $KLMNPQ$. Therefore the area of this hexagon is 6 times bigger than the area of the triangle LOK . In this triangle $LO = KO$ (O is the center of the hexagon), $\angle LOK = 360/6 = 60^\circ$ so LOK is an equilateral triangle, and its area is $KL^2\sqrt{3}/4$. Substituting KL with its value found above $KL = \sqrt{2}/2$, we obtain $A(LOK) = \sqrt{3}/8$, and $A(KLMNPQ) = 6\sqrt{3}/8 = 3\sqrt{3}/4$.



6. For every 8-digit positive whole number $abcdefgh$ (a, b, c, d, e, f, g, h – its digits, $a > 0$) let's define its “beauty” as the value of the expression $a - b + c - d + e - f + g - h$ (for example, number 20080316 has a “beauty” -14.) Find an average “beauty” of all 8-digit positive whole numbers.

Answer: 0.5

Solution: An average “beauty” of all 8-digit positive whole numbers (AB) is equal to the total “beauty” of all 8-digit positive whole numbers (TB), divided by the total number of 8-digit positive whole numbers (T). To find the value of TB , we need to add all “beauties” for all 8-digit positive whole numbers. But we can do in using the following technique: first, we will add all first (from the left) digits for all 8-digit whole numbers, then we will subtract all their second digits, then we will add all their third digits, and so force. Since all digits from 1 to 9 are equally distributed as first digits of T 8-digit positive whole numbers, each digit from 1 to 9 would be added $T/9$ times as first digit, so the sum of all first digits of all 8-digit positive whole numbers would be $(1 + 2 + \dots + 9) \times T/9 = 5T$. Since all digits from 0 to 9 are equally distributed as second digits of T 8-digit positive whole numbers, each digit from 0 to 9 would be subtracted $T/10$ times as second digit, so the sum of all second digits of all 8-digit positive whole numbers would be $(0 + 1 + 2 + \dots + 9) \times T/10 = 4.5T$, and this number would be subtracted from $5T$. Then we would add the sum of all third digits of all 8-digit positive whole numbers ($4.5T$, similar to the sum of their second digits), subtract the sum of all fourth digits of all 8-digit

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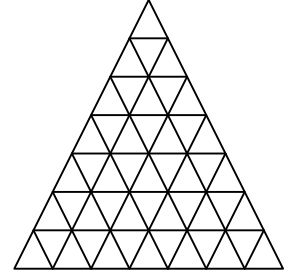
positive whole numbers ($4.5T$, similar to the sum of their second digits), and so force: $TB = 5T - 4.5T + 4.5T - 4.5T + 4.5T - 4.5T + 4.5T - 4.5T = 0.5T$. Therefore, $AB = TB/T = 0.5T/T = 0.5$.

7. How many parallelograms are in this figure? Count all parallelograms you can see without drawing additional lines. All lines shown are parallel to the sides of the big triangle.

Answer: 378

Solution: We assume that length of each side of the big triangle is 7. For each of the possible parallelograms, its sides are parallel to exactly two sides of the big triangle. There are exactly 3 ways to select 2 sides of the triangle, so we will count all parallelograms with sides parallel to non-horizontal sides of the big triangle, and multiply the result by 3. Each of such parallelograms is “inscribed” in exactly one triangle that has the same orientation as the big triangle, shares its top vertex with the parallelogram’s top vertex, and has its bottom (horizontal) side containing the parallelogram’s bottom vertex. A triangle with side N has exactly $N - 1$ parallelograms “inscribed” in it, so we first count the number of triangles by their size. The triangle sides could have length 7 (1 triangle), length 6 (1 + 2 = 3 triangles), length 5 (1 + 2 + 3 = 6 triangles), length 4 (1 + 2 + 3 + 4 = 10 triangles), length 3 (1 + 2 + 3 + 4 + 5 = 15 triangles), length 2 (1 + 2 + 3 + 4 + 5 + 6 = 21 triangles), and length 1 (1 + 2 + 3 + 4 + 5 + 6 + 7 = 28 triangles). Now we can count all the parallelograms “inscribed” in all these triangles:

$1 \times 6 + 3 \times 5 + 6 \times 4 + 10 \times 3 + 15 \times 2 + 21 \times 1 + 28 \times 0 = 6 + 15 + 24 + 30 + 30 + 21 = 126$ parallelograms. Therefore, there are $3 \times 126 = 378$ parallelograms in the figure shown.



8. There are 10 coins that look the same, each of them in its own clear plastic bag (bags are labeled using letters $A, B, C, D, E, F, G, H, I, J$ and have the same weight.) 5 of these coins are genuine, having the same weight, and 5 others are fake, also having the same weight, but a fake coin is lighter than a genuine one. There is also a correctly functioning balance without any reference weights and without a scale. An attorney knows (from his client) which 5 coins are fake ones. But he cannot identify (directly or indirectly) any of fake coins to a jury, otherwise his client would be punished. Therefore the attorney cannot make weigh attempts that would lead to logical identification of at least one of the fake coins. To defend the client, the attorney has to split (with an explanation that satisfies the jury) the given 10 coins into 5 pairs having one genuine and one fake coins each. What is the least number of weigh attempts the attorney should make to surely reach this goal? The attorney does not have any additional coins.

Answer: 2

Solution: It is obvious that the attorney should not put different number of coins on the balance pans, otherwise the jury would not get any useful information. If the attorney puts exactly 1, 2, 3, or 5 coins on each of the balance pans, it would not be sufficient to surely identify at least one pair of one genuine and one fake coins regardless of the result of this weigh attempt. If the attorney puts exactly 4 coins on each of the balance pans, and pans would be in balance, the jury could conclude that these 8 coins contain even number of genuine coins and even number of fake coins, therefore remaining 2 coins contain one genuine coin and one fake coin, but no more pairs could be identified without additional weigh attempts. If the attorney puts exactly 4 coins on each of the balance pans, and pans

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would not be in balance, it would not be sufficient to surely identify at least one pair of one genuine and one fake coins. So, to surely reach his goal, the attorney has to make at least 2 weigh attempts.

Let's show how the attorney could surely split the given 10 coins into 5 pairs having one genuine and one fake coin each using 2 weigh attempts. Let's suppose that coins A, B, C, D, E are genuine, coins F, G, H, I, J are fake, the attorney knows that, but the jury does not.

Attempt 1: $ABFG = CDHI$ – the jury concludes that pair EJ contains 1 genuine and 1 fake coins, and sets $ABFG$ and $CDHI$ contain 2 genuine and 2 fake coins each

Attempt 2: $AEFJ = BCGH$ – the jury concludes, that pair DI contains 1 genuine and 1 fake coins, sets $AEFJ$ and $BCGH$ contain 2 genuine and 2 fake coins each, therefore pair CH contains 1 genuine and 1 fake coins, therefore pair BG contains 1 genuine and 1 fake coins, therefore pair AF contains 1 genuine and 1 fake coins.

If the attorney uses the above strategy, the jury cannot surely identify any fake coin. Indeed, in each pair any of the coins could be fake one. If coins A, B, C, D, E are genuine, coins F, G, H, I, J are fake, in each pair coin with the “smaller” letter (based on letters' ordinal numbers in the alphabet) would be genuine. But if coins A, B, C, D, E are fake, coins F, G, H, I, J are genuine, the results of both weigh attempts above would not change, and in each pair coin with the “smaller” letter would be fake.

9th – 10th Grades

1. How many consecutive zeros are there at the end of the number $\frac{30!}{4 \times 5!}$?

Answer: 6

Solution: Since $30! = 1 \times 2 \times 3 \times \dots \times 29 \times 30$, and the only factors divisible by 5 are 5, $10 = 2 \times 5$, $15 = 3 \times 5$, $20 = 2 \times 2 \times 5$, $25 = 5 \times 5$, $30 = 2 \times 3 \times 5$, 2 and 3 are prime numbers, different from 5, we can conclude that the number $30!$ has factor 5 exactly 7 times in its prime decomposition. And it has factor 2 at least 15 times in its prime decomposition, because there are 15 even factors in $1 \times 2 \times 3 \times \dots \times 29 \times 30$, and similarly it has factor 3 at least 10 times. On the other hand, $4 \times 5! = 2^5 \times 3 \times 5$, so it has factor 2 exactly 5 times, factor 3 exactly once and factor 5 exactly once in its prime decomposition. Therefore the number $\frac{30!}{4 \times 5!}$ is a positive whole number that has factor 5 exactly 6 times and factor 2 at least 10 times in its prime decomposition. So it is divisible by 10^6 ($10 = 2 \times 5$), but not divisible by 10^7 (there are not enough factors 5). So there are exactly 6 zeros at the end of the number $\frac{30!}{4 \times 5!}$.

2. The numbers $\sqrt[3]{3}$, $\sqrt[4]{3}$, and x are respectively first, second and third terms of a geometric sequence. If x is in the form 3^y , find y .

Answer: $3/8$

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Solution: Let's use the definition of a geometric sequence: $\frac{\sqrt[4]{3}}{\sqrt[8]{3}} = \frac{x}{\sqrt[4]{3}}$, $\sqrt[4]{3} \times \sqrt[4]{3} = x \times \sqrt[8]{3}$,
 $x = \frac{\sqrt[4]{3} \times \sqrt[4]{3}}{\sqrt[8]{3}} = 3^{\frac{1}{4} + \frac{1}{4} - \frac{1}{8}} = 3^{\frac{3}{8}}$, but $x = 3^y$, therefore $y = 3/8$.

3. The increasing sequence 2, 3, 5, 6, 7, 10, ... consists of all natural numbers which are neither perfect squares nor perfect cubes. Find the 75th term of this sequence.

Answer: 86

Solution: Let's start from the number 75. It is neither perfect cube nor perfect square, so it belongs to our sequence S . In the increasing sequence of all natural numbers there are exactly 8 perfect squares ($1^2, 2^2, \dots, 8^2 = 64 < 75, 9^2 = 81 > 75$) and 4 perfect cubes ($1^3, 2^3, 3^3, 4^3 = 64 < 75, 5^3 = 125 > 75$) less than 75, but since 1 and 64 (and only those numbers between 1 and 75) are both perfect squares and perfect cubes, there are exactly 10 natural numbers less than 75 that are not included in S . At this point it looks like we just need to add 10 to 75 and say that the number 85 is the 75th term of the sequence S , but this would not be true! There is exactly one perfect square (81) and no perfect cubes between 75 and 85, so we need to add 1 to 85 to conclude that then number 86 is the 75th term of the sequence S , and this would be true since 86 itself belongs to S , and no further adjustments are required.

Alternate Solution: We can just write down all natural numbers from 1 to 100 in increasing order, cross out all of them that are either perfect squares (10 numbers) or perfect cubes (4 numbers, 2 of them are also perfect squares), and just find (by counting) the 75th term of the given sequence.

4. Let ABC be a triangle. Suppose that $AB = x + 4$, $BC = x + 7$, and $AC = 4x$. For angle CAB to be the only largest angle of the triangle, x must satisfy condition $m < x < n$. Find the sum of the least possible value of n and the most possible value of m .

Answer: $3\frac{1}{12}$

Solution: Let's express all the conditions using inequalities. All triangle sides should be positive: $x + 4 > 0$, $x + 7 > 0$, $4x > 0$; BC should be strictly longer than any of the other two sides (since opposite angle CAB is the only largest angle of the triangle): $x + 7 > x + 4$, $x + 7 > 4x$; and the longest side (BC) should be shorter than the total length of the other two sides (triangle inequality): $x + 7 < x + 4 + 4x$. After simplifying we get $x > -4$, $x > -7$, $x > 0$, $7 > 4x$, $x < 7/3$, $x > 3/4$. After removing redundant conditions, we get $3/4 < x < 7/3$, and this inequality is equivalent to the problem condition. Therefore the least possible value of n is $7/3$, and the most possible value of m is $3/4$, so their sum is $\frac{7}{3} + \frac{3}{4} = \frac{37}{12} = 3\frac{1}{12}$.

5. Jessie has some 7ϕ , some 13ϕ , and some 37ϕ stamps. She has the same number of two kinds of these stamps and a different number of the third kind. The total value of all her stamps is \$2.49. How many of each kind of stamps does she have?

Answer: $(7 \times 7\phi, 4 \times 13\phi, 4 \times 37\phi)$, $(3 \times 7\phi, 9 \times 13\phi, 3 \times 37\phi)$

Solution: Let's consider 3 separate cases.

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Case 1: Jessie has the same number (x) of 7¢ and 13¢ stamps, and a different number (y) of 37¢ stamps, x and y are different non-negative whole numbers. Therefore $(7 + 13)x + 37y = 249$, $20x + 37y = 249$. $7y$ should have 9 as its units digit, therefore y should have 7 as its units digit, but $37 \times 7 = 259 > 249$, so this case is impossible.

Case 2: Jessie has the same number (x) of 7¢ and 37¢ stamps, and a different number (y) of 13¢ stamps, x and y are different non-negative whole numbers. Therefore $(7 + 37)x + 13y = 249$, $44x + 13y = 249$. Since $44 \times 6 = 264 > 249$, x could only be 0, 1, 2, 3, 4, or 5, and $249 - 44x$ should be divisible by 13. 249 is not divisible by 13, $249 - 44 = 205$ is not divisible by 13, $205 - 44 = 161$ is not divisible by 13, $161 - 44 = 117 = 9 \times 13$, $117 - 44 = 73$ is not divisible by 13, $73 - 44 = 29$ is not divisible by 13, $29 - 44 < 0$. So, the only possible solution in this case is $x = 3$, $y = 9$.

Case 3: Jessie has the same number (x) of 13¢ and 37¢ stamps, and a different number (y) of 7¢ stamps, x and y are different non-negative whole numbers. Therefore $(13 + 37)x + 7y = 249$, $50x + 7y = 249$. Since $50 \times 5 = 250 > 249$, x could only be 0, 1, 2, 3, or 4, and $249 - 50x$ should be divisible by 7. 249 is not divisible by 7, $249 - 50 = 199$ is not divisible by 7, $199 - 50 = 149$ is not divisible by 7, $149 - 50 = 99$ is not divisible by 7, $99 - 50 = 49 = 7 \times 7$, $49 - 50 < 0$. So, the only possible solution in this case is $x = 4$, $y = 7$.

6. Let's define an 8-digit positive whole number $abcdefgh$ (a, b, c, d, e, f, g, h – its digits, $a > 0$) as “beautiful” if number $aceg$ is greater than number $bdfh$ (for example, 20080316.) Find the number of “beautiful” numbers.

Answer: 49495500

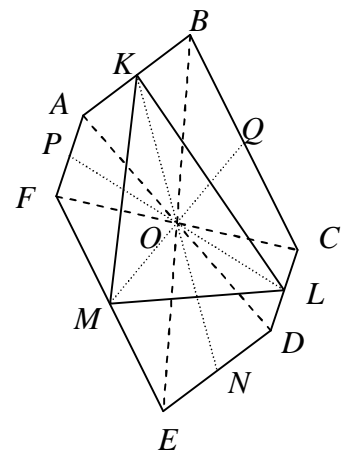
Solution: In an 8-digit positive whole number $abcdefgh$ number $aceg$ could be any 4-digit positive whole number from 1000 till 9999 ($9999 - 999 = 9000$ choices). For any $aceg$ between 1000 and 9999, number $bdfh$ could be any whole number between 0 (written as 0000) and $aceg - 1$ inclusive to satisfy condition “number $aceg$ is greater than number $bdfh$ ”. Now we can count (by using the so-called “addition rule”) that there are

$1000 + 1001 + 1002 + \dots + 9998 + 9999 = (1000 + 9999) \times 9000 / 2 = 49495500$ “beautiful” numbers.

7. In a convex hexagon $ABCDEF$ quadrilaterals $ABDE$, $BCEF$, and $CDFA$ are parallelograms, K , L , and M are the midpoints of AB , CD , and EF respectively. Find the ratio of the area of the triangle KLM to the area of the hexagon $ABCDEF$.

Answer: 3:8

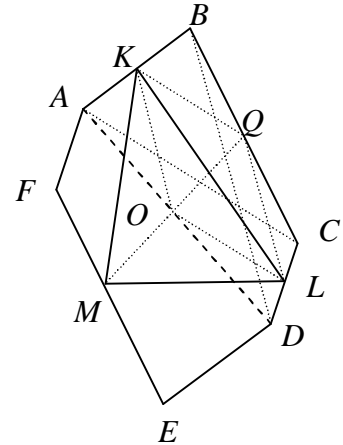
Solution: Since $ABDE$ is a parallelogram, its diagonals AD and BE intersect at their common midpoint O . Since $BCEF$ is a parallelogram, its diagonals BE and CF intersect at their common midpoint O (it is the same as midpoint of BE). Therefore the point O is a center of symmetry of the convex hexagon $ABCDEF$. Let N , P , Q be the midpoints of DE , FA , and BC respectively. Since the point O is the center of symmetry of the parallelogram $ABDE$, O is the midpoint of KN . $\angle LKM < 180^\circ$ ($ABCDEF$ is a complex hexagon). The point N is between L and M on the broken line $LDEM$, therefore the segment KN is inside the angle



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LKM , and so the center O . Similarly the center O is inside the angles KLM and KML , so it is inside the triangle KLM .

Let's consider convex quadrilateral $ABCD$. The points K, Q, L , and O are the midpoints of its sides, therefore $KQ \parallel AC$, $KQ = AC/2$, $OL \parallel AC$, $OL = AC/2$, $KO \parallel BD$, $KO = BD/2$, $QL \parallel BD$, $QL = BD/2$. The area of the triangle KOL is equal $OK \times OL \times \sin(\angle KOL)/2$. The area of the quadrilateral $ABCD$ is equal to $AC \times BD \times \sin(\text{the angle between } AC \text{ and } BD)/2 = (2 \times OL) \times (2 \times OK) \times \sin(\angle KOL)/2$, that is 4 times bigger than the area of the triangle KOL (the angle between AC and BD is equal either to $\angle KOL$ or to $180^\circ - \angle KOL$, in any case $\sin(\text{that angle}) = \sin(\angle KOL)$). On the other hand, since the point O is a center of symmetry of the convex hexagon $ABCDEF$, the area of the quadrilateral $ABCD$ is 2 times smaller than the area of the entire hexagon $ABCDEF$. So the area of the triangle KOL (and similarly the area of each of the triangles LOM and KOM) is 8 times smaller than the area of the entire hexagon. But the area of the triangle KLM is equal to the sum of the areas of the triangles KOL , LOM , and KOM , so the area of the triangle KLM is equal to the area of the entire hexagon multiplied by $3/8$.



8. There are 10 coins that look the same, each of them in its own clear plastic bag (bags are labeled using letters $A, B, C, D, E, F, G, H, I, J$ and have the same weight.) 5 of these coins are genuine, having the same weight, and 5 others are fake, also having the same weight, but a fake coin is lighter than a genuine one. There is also a correctly functioning balance without any reference weights and without a scale. An attorney knows (from his client) which 5 coins are fake ones. But he cannot identify (directly or indirectly) any of fake coins to a jury, otherwise his client would be punished. Therefore the attorney cannot make weigh attempts that would lead to logical identification of at least one of the fake coins. To defend the client, the attorney has to identify (with an explanation that satisfies the jury) any genuine coin obeying the following restriction: each time he has to place exactly two coins on each (left and right) balance pans. What is the least number of weigh attempts the attorney should make to surely reach this goal? The attorney does not have any additional coins.

Answer: 2

Solution: If the attorney makes just one weigh attempt, it obviously would not be sufficient to surely identify any genuine coin regardless of the result of this weigh attempt. So, to surely reach his goal, the attorney has to make at least 2 weigh attempts.

Let's show how the attorney could surely identify any genuine coin to the jury using 2 weigh attempts. Let's suppose that coins A, B, C, D, E are genuine, coins F, G, H, I, J are fake, the attorney knows that, but the jury does not.

Attempt 1: $AB > FG$

Attempt 2: $AF > HI$

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From the above outcomes, the jury can conclude that the coin A is genuine. Indeed, let's suppose that the coin A is fake. Then both coins F and G should also be fake ones (Outcome 1), but in this case Outcome 2 is impossible. So, the coin A is a genuine one.

If the attorney uses the above strategy, the jury cannot surely identify any fake coin. None of the coins B , C , D , E can be surely identified as fake one since all of them could be genuine (as per the assumption above). The coin F (or G or J) also could be genuine (if the coin E is fake), and the results of both weigh attempts above would not change. The coin H (or I) also could be genuine (if the coin F is genuine, and both coins C and D are fake), and the results of both weigh attempts above would not change.

11th – 12th Grades

1. Find the smallest positive whole number that ends with 17, is divisible by 17, and the sum of its digits is 17.

Answer: 15317

Solution: Any positive whole number that ends with 17 could be written in the form $\underline{X17}$ ($\underline{X17} = 100 \times X + 17$) where X is a non-negative whole number. This number is divisible by 17 if and only if $100X$ is divisible by 17 (since 17 is divisible by 17), that is the same as X is divisible by 17 (since 100 and 17 are relatively prime numbers). The sum of all the digits of the number $\underline{X17}$ is equal to 8 ($1 + 7 = 8$) added to the sum of all the digits of the number X . So the sum of all the digits of the number $\underline{X17}$ is equal to 17 if and only if the sum of all the digits of the number X is equal to 9. Therefore X has to be divisible by 9, and it is divisible by 17. Since 9 and 17 are relatively prime numbers, X has to be divisible by $9 \times 17 = 153$. The smallest non-negative number divisible by 153 is 0, but the sum of all its digits is equal to 0, not 9. The next smallest non-negative number divisible by 153 is 153 itself, the sum of all its digits is equal to 9, so the smallest possible number X satisfying our conditions is 153. But the smaller X the smaller $\underline{X17}$ (and vice versa) therefore the smallest positive whole number that ends with 17, is divisible by 17, and the sum of its digits is 17, would be 15317.

2. The numbers $\sqrt[8]{3}$, $\sqrt[4]{3}$, and x are respectively first, second and third terms of a geometric sequence. If x is in the form 3^y , find y .

Answer: $3/8$

Solution: Let's use the definition of a geometric sequence: $\frac{\sqrt[4]{3}}{\sqrt[8]{3}} = \frac{x}{\sqrt[4]{3}}$, $\sqrt[4]{3} \times \sqrt[4]{3} = x \times \sqrt[8]{3}$,

$$x = \frac{\sqrt[4]{3} \times \sqrt[4]{3}}{\sqrt[8]{3}} = 3^{\frac{1}{4} + \frac{1}{4} - \frac{1}{8}} = 3^{\frac{3}{8}}, \text{ but } x = 3^y, \text{ therefore } y = 3/8.$$

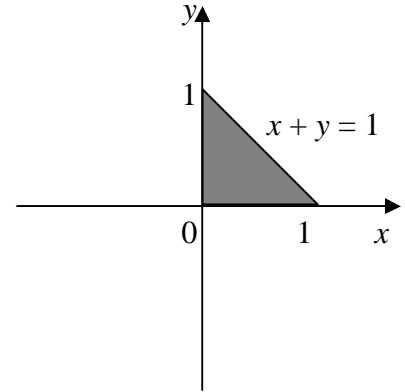
3. What is the area of the region in the Cartesian Plane that is defined by the following inequality: $|x| + |y| + |x + y| \leq 2$?

Answer: 3

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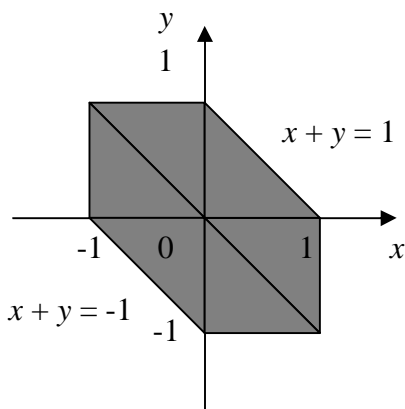
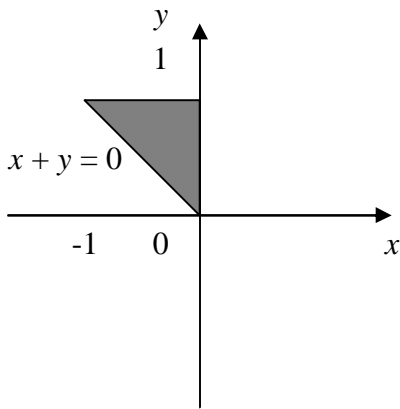
Solution: To draw the region defined by the inequality, let's consider several cases.

Case 1: $x \geq 0$, $y \geq 0$, therefore $x + y \geq 0$, $|x| = x$, $|y| = y$, $|x + y| = x + y$, and we can simplify the original inequality to the form $x + y \leq 1$. The sub-region in the Cartesian Plane that corresponds to this case is the triangle bounded by the straight lines $x = 0$, $y = 0$, $x + y = 1$.



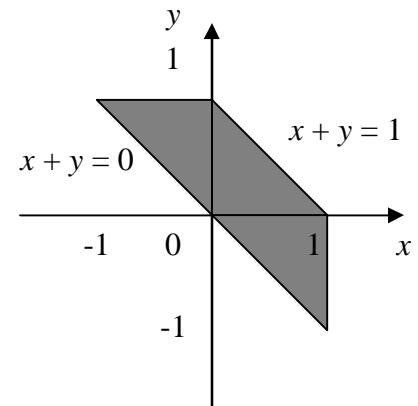
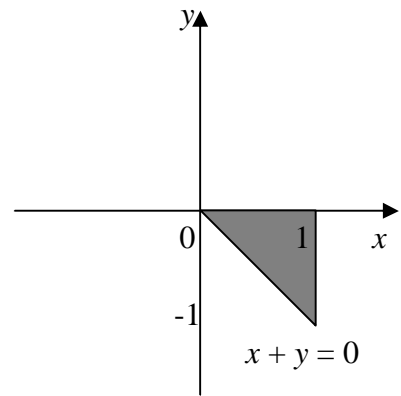
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Case 2: $x \geq 0, y \leq 0, x + y \geq 0$, therefore, $|x| = x, |y| = -y, |x + y| = x + y$, and we can simplify the original inequality to the form $x \leq 1$. The sub-region in the Cartesian Plane that corresponds to this case is the triangle bounded by the straight lines $x = 1, y = 0, x + y = 0$.



Since the original inequality would not change if we transpose x and y , the line $x = y$ would be the line of symmetry of the corresponding region in the Cartesian Plane. Therefore we can just reflect the sub-region found in the Case 2 over the line $x = y$ instead of considering the corresponding case ($x \leq 0, y \geq 0, x + y \geq 0$).

Since $|-z| = |z|$ for any z , the original inequality would not change if we replace x with $-x$ and y with $-y$, so the origin would be the center of symmetry of the corresponding region in the Cartesian Plane. Therefore we can just rotate all the sub-regions found above by 180° (in any direction) around the origin instead of considering the other 3 cases.



Finally we see that the region in the Cartesian Plane that is defined by the original inequality is a hexagon, that is made of 2 squares 1×1 and two right isosceles triangles that could be put together to make up another square 1×1 . Therefore, the area of this

region is equal to $3 \times 1^2 = 3$.

4. How many consecutive zeros are there at the end of the number $9^{999} + 1$?

Answer: 1

Solution: Let's consider the sequence of the tens and units digits of numbers 9^n , where n is a non-negative integer. The first few terms are 01 ($n = 0$), 09 ($n = 1$), 81 ($n = 2$), 29 ($n = 3$), 61 ($n = 4$), 49 ($n = 5$), 41 ($n = 6$), 69 ($n = 7$), 21 ($n = 8$), 89 ($n = 9$), 01 ($n = 10$). Since the tens and units digits of the number $9^{n+1} = 9 \times 9^n$ depend only on the tens and unit digits of the number 9^n , this sequence is a periodic one with the period 10 (01 corresponds to $n = 0$ and $n = 10$). Therefore the tens and unit digits of the number 9^{999} are the same as the tens and units digits of the number 9^9 (i.e. 89). So the tens and units digits of the number $9^{999} + 1$ are 90, and there is exactly one zero at the end of this number.

Alternate Solution: Let's mention that any number 9^n (n is a positive integer) has either 1 (for even n) or 9 (for odd n) as its units digit. Therefore when we multiply this number by 9 to get the next

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power of 9, either 0 ($1 \times 9 = 09$) or 8 ($9 \times 9 = 81$) would need to be carried to the tens place. Now it is easy to prove by induction (do it!) that any number 9^n (n is a positive integer) has an even digit as its tens digit. 999 is an odd number, so the units digit of the number 9^{999} would be 9, and its tens digit would be even. Therefore there is exactly one zero at the end of the number $9^{999} + 1$.

5. What is the coefficient of x^{99} in the expansion of $(x-1)(x-2)\dots(x-100)$?

Answer: -5050

Solution: In order to get x^{99} in the expansion of $(x-1)(x-2)\dots(x-100)$, we would have to take a term x from 99 parentheses, a numerical term $(-1, -2, \dots, -100)$ from the remaining parentheses, multiply all these terms, and then add all such expressions together. Therefore, the coefficient of x^{99} in the expansion of $(x-1)(x-2)\dots(x-100)$ would be equal to the sum

$$(-1) + (-2) + \dots + (-100) = -(1 + 2 + \dots + 100) = -(1 + 100) \times 100 / 2 = -101 \times 50 = -5050.$$

6. Let's define an 8-digit positive whole number $abcdefgh$ (a, b, c, d, e, f, g, h – its digits, $a > 0$) as “beautiful” if $a - b + c - d + e - f + g - h > 0$ (for example, 20089123.) Find the number of “beautiful” numbers.

Answer: 45000000

Solution: For every 8-digit positive whole number $X = abcdefgh$ (a, b, c, d, e, f, g, h – its digits, $a > 0$) let's define its “beauty” $B(X)$ as the value of the expression $a - b + c - d + e - f + g - h$ (for example, number 20080316 has a “beauty” -14), then the number X would be “beautiful” if and only if its “beauty” is positive. Let's also consider an 8-digit positive whole number $Y = ijklmnop$, where $i = 10 - a, j = 9 - b, k = 9 - c, \dots, p = 9 - h, i, j, k, l, m, n, o, p$ – its digits, $i > 0$ (actually, $Y = 109999999 - X$). Let's find its “beauty” $B(Y)$: $B(Y) = i - j + k - l + m - n + o - p$, therefore $B(Y) = (10 - a) - (9 - b) + (9 - c) - (9 - d) + (9 - e) - (9 - f) + (9 - g) - (9 - h) = 1 - (a - b + c - d + e - f + g - h)$, so $B(Y) = 1 - B(X)$. Therefore $B(X) + B(Y) = 1$, both numbers $B(X)$ and $B(Y)$ are integers, so exactly one of them is positive. So exactly one of the two numbers X and Y is “beautiful”. Now we can group all 90000000 8-digit positive whole numbers ($90000000 = 99999999 - 9999999$) into pairs (10000000, 99999999), (10000001, 99999998), ..., (54444449, 55555550), where each pair contains two numbers that add up to 109999999 (since 109999999 is an odd number, none of 8-digit positive whole numbers would be paired to itself). As shown above, each pair would contain exactly one “beautiful” number, therefore the number of “beautiful” numbers would be the same as the number of pairs that is equal to $90000000 / 2 = 45000000$.

7. In a convex hexagon $ABCDEF$ for each pair of its opposite sides one of these sides (let's call it “long”) is parallel to and 50% longer than another one (let's call it “short”.) $K, L,$ and M are the midpoints of the hexagon's “short” sides. Find the ratio of the area of the triangle KLM to the area of the hexagon $ABCDEF$.

Answer: 16:37

Solution: First, let's prove that in the given hexagon its “short” and “long” sides are alternating, in other words, “short” sides do not share a hexagon's vertex. Indeed, if both AB and BC would be “short” sides, let's consider opposite sides FA and CD . One of them is also a “short” side, let it be CD .

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Let's consider vectors $\vec{x} = \overrightarrow{AB}$, $\vec{y} = \overrightarrow{BC}$, $\vec{z} = \overrightarrow{CD}$, then $\overrightarrow{DE} = -1.5\vec{x}$, $\overrightarrow{EF} = -1.5\vec{y}$, $\overrightarrow{FA} = -1.5\vec{z}$. In any hexagon $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} + \overrightarrow{EF} + \overrightarrow{FA} = \vec{0}$, $\vec{x} + \vec{y} + \vec{z} - 1.5\vec{x} - 1.5\vec{y} - 1.5\vec{z} = \vec{0}$, $0.5(\vec{x} + \vec{y} + \vec{z}) = \vec{0}$, $\vec{x} + \vec{y} + \vec{z} = \vec{0}$, $\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} = \vec{0}$, therefore $A = D$, but this contradicts to the convexity of the given hexagon.

Now let's suppose that AB , CD and EF are "short" sides, the other 3 sides are "long", K , L , and M are the midpoints of the sides AB , CD , and EF respectively.

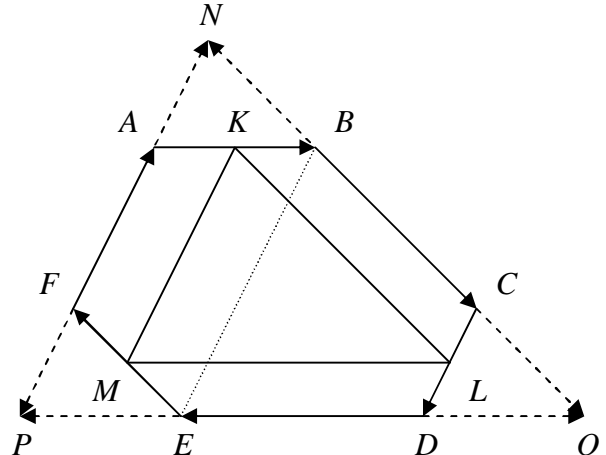
Let's consider vectors $\vec{x} = \overrightarrow{AB}$, $\vec{y} = \overrightarrow{CD}$, $\vec{z} = \overrightarrow{EF}$, then $\overrightarrow{DE} = -1.5\vec{x}$, $\overrightarrow{FA} = -1.5\vec{y}$, $\overrightarrow{BC} = -1.5\vec{z}$. Again,

$$\begin{aligned} \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} + \overrightarrow{EF} + \overrightarrow{FA} &= \vec{0} \\ \vec{x} - 1.5\vec{z} + \vec{y} - 1.5\vec{x} + \vec{z} - 1.5\vec{y} &= \vec{0}, \quad 0.5(\vec{x} + \vec{y} + \vec{z}) = \vec{0} \\ \vec{x} + \vec{y} + \vec{z} &= \vec{0}. \end{aligned}$$

Let's draw a point N , $\overrightarrow{BN} = \vec{z}$, then $\overrightarrow{AN} = \overrightarrow{AB} + \overrightarrow{BN} = \vec{x} + \vec{z} = -\vec{y}$. Since $\overrightarrow{BN} = \vec{z}$, $\overrightarrow{BC} = -1.5\vec{z}$, the point B is located between points N

and C . Since $\overrightarrow{AN} = -\vec{y}$, $\overrightarrow{AF} = 1.5\vec{y}$, the point A is located between points N and F . Similarly we will draw points O ($\overrightarrow{DO} = \vec{x}$, $\overrightarrow{CO} = -\vec{z}$, the point C is located between points B and O , the point D is located between points E and O) and P ($\overrightarrow{FP} = \vec{y}$, $\overrightarrow{EP} = -\vec{x}$, the point E is located between points D and P , the point F is located between points A and P).

As a result, we get the triangle NOP , and vertices of the hexagon $ABCDEF$ divide all its sides in the ratio 1:1.5:1, that is the same as 2:3:2. Let T be the area of the triangle NOP . The triangle NBA is similar to the triangle NOP ($NB:NO = NA:NP = 2:7$), therefore the area of the triangle NBA (and similarly the area of each of the triangles COD and FEP) is equal to $(2/7)^2 T = 4T/49$, and the area of the hexagon $ABCDEF$ is equal to $T - 3 \times (4T/49) = 37T/49$. The triangle BOE is homothetic to the triangle NOP ($BO:NO = OE:OP = 5:7$), therefore the coefficient of homothety is equal to $5/7$, $BE \parallel NP$, the quadrilateral $ABEF$ is a trapezoid (AB is not parallel to EF since $AB \parallel DE$), $BE = 5NP/7$, $AF = 3NP/7$. But the points K and M are the midpoints of the sides AB and EF respectively, therefore $KM = (AF + BE)/2 = 4NP/7$. Similarly, $KL = 4NO/7$ and $LM = 4OP/7$, so the triangle KLM is similar to the triangle NOP (with the coefficient of similarity $4/7$), and the area of the triangle KLM is equal to $(4/7)^2 T = 16T/49$. Finally, the ratio of the area of the triangle KLM to the area of the hexagon $ABCDEF$ is equal to $16T/49 : 37T/49 = 16:37$.



8. There are 10 coins that look the same, each of them in its own clear plastic bag (bags are labeled using letters $A, B, C, D, E, F, G, H, I, J$ and have the same weight.) 5 of these coins are genuine, having the same weight, and 5 others are fake, also having the same weight, but a fake coin is lighter than a genuine one. There is also a correctly functioning balance without any reference weights and without a scale. An attorney knows (from his client) which 5 coins are fake ones. But he cannot identify (directly or indirectly) any of fake coins to a jury, otherwise his client would be punished. Therefore the attorney cannot make weigh attempts that would lead to logical identification of at least one of the fake coins. To defend the client, the attorney has to identify (with an explanation that

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satisfies the jury) some genuine coins (the more the better) using just 3 weigh attempts and obeying the following restriction: each time he has to place exactly two coins on each (left and right) balance pans. What is the most number of genuine coins the attorney could identify to the jury? The attorney does not have any additional coins.

Answer: 4

Solution: It is obvious that the attorney cannot identify all 5 genuine coins to the jury, otherwise the jury could conclude that all 5 remaining coins are fake ones. So the attorney could identify no more than 4 genuine coins to the jury.

Let's show how the attorney could surely identify 4 genuine coins to the jury using 3 weigh attempts. Let's suppose that coins A, B, C, D, E are genuine, coins F, G, H, I, J are fake, the attorney knows that, but the jury does not.

Attempt 1: $AB > FG$

Attempt 2: $AB > HI$

Attempt 3: $AB = CD$

From the above outcomes, the jury can conclude that all the coins $A, B, C,$ and D are genuine. Indeed, let's suppose that at least one of these 4 coins is a fake one. Then at least one of the coins A and B and at least one of the coins C and D should be fake (Outcome 3), all the coins F and G (Outcome 1) and H and I (Outcome 2) should also be fake one, so in total there should be at least 6 fake coins, but there are just 5 fake coins! So, all the coins $A, B, C,$ and D are genuine ones.

If the attorney uses the above strategy, the jury cannot surely identify any fake coin. None of the coins E, F, G, H, I, J can be surely identified as fake one since any of them could be genuine (the coin E could be genuine as per the assumption above, and any of the other 5 coins could be genuine instead of the coin E , and the results of all three weigh attempts above would not change).